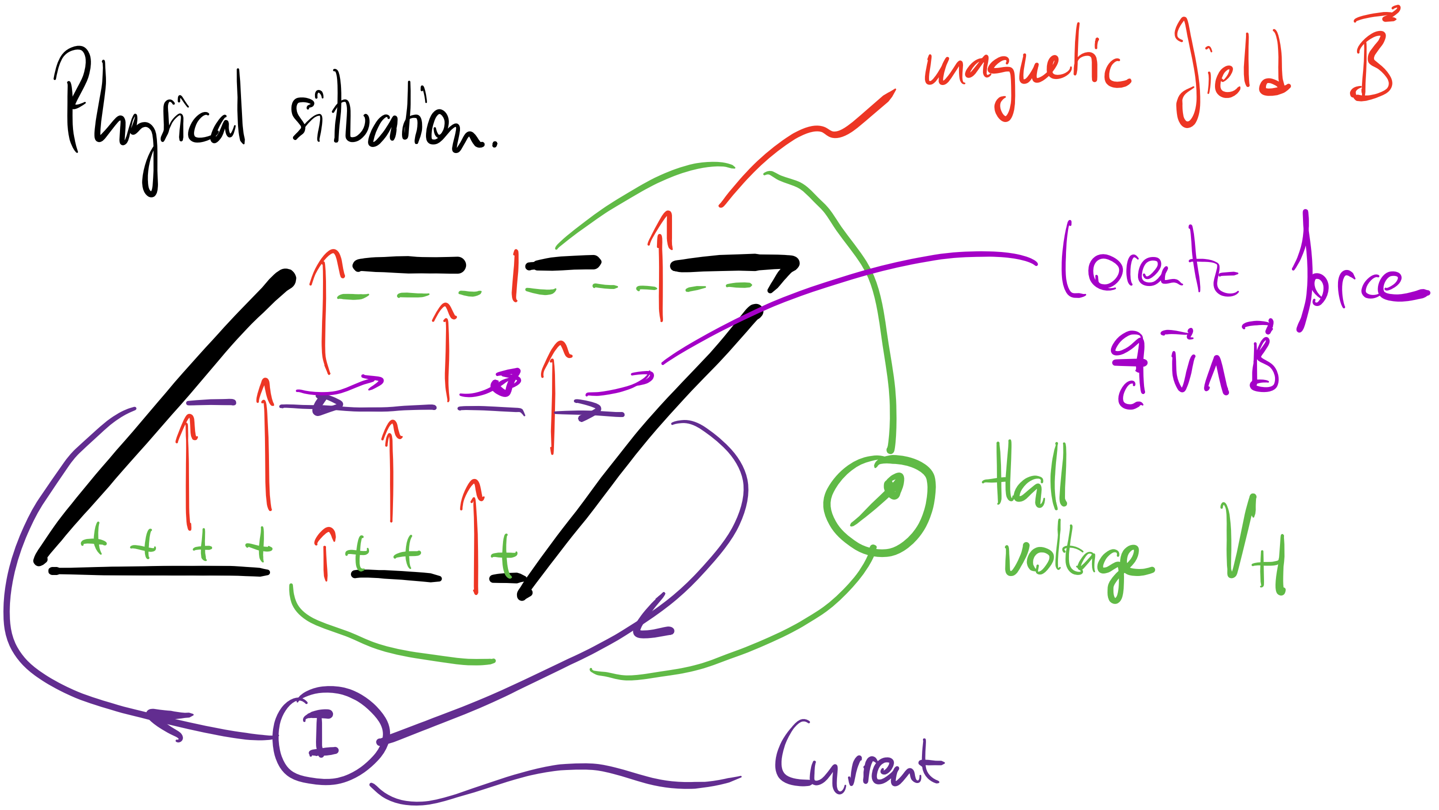
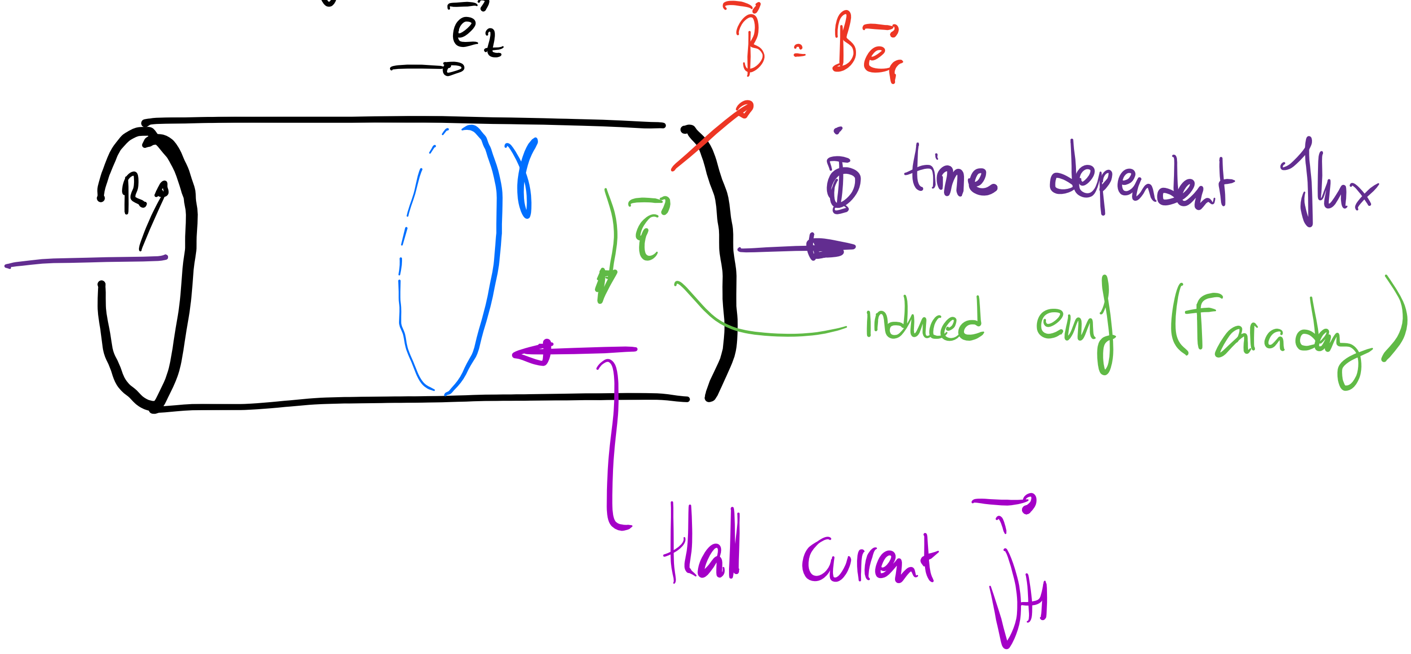
Aspects of topological quantum transport

Sven Bachmann, August 2022.

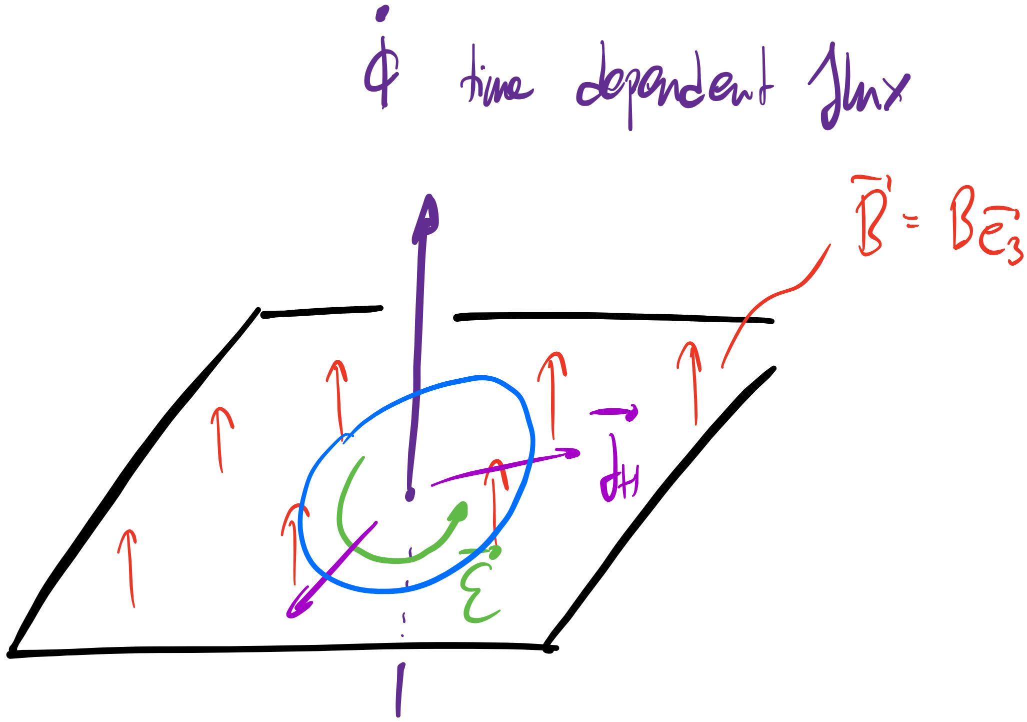
Physical situation.



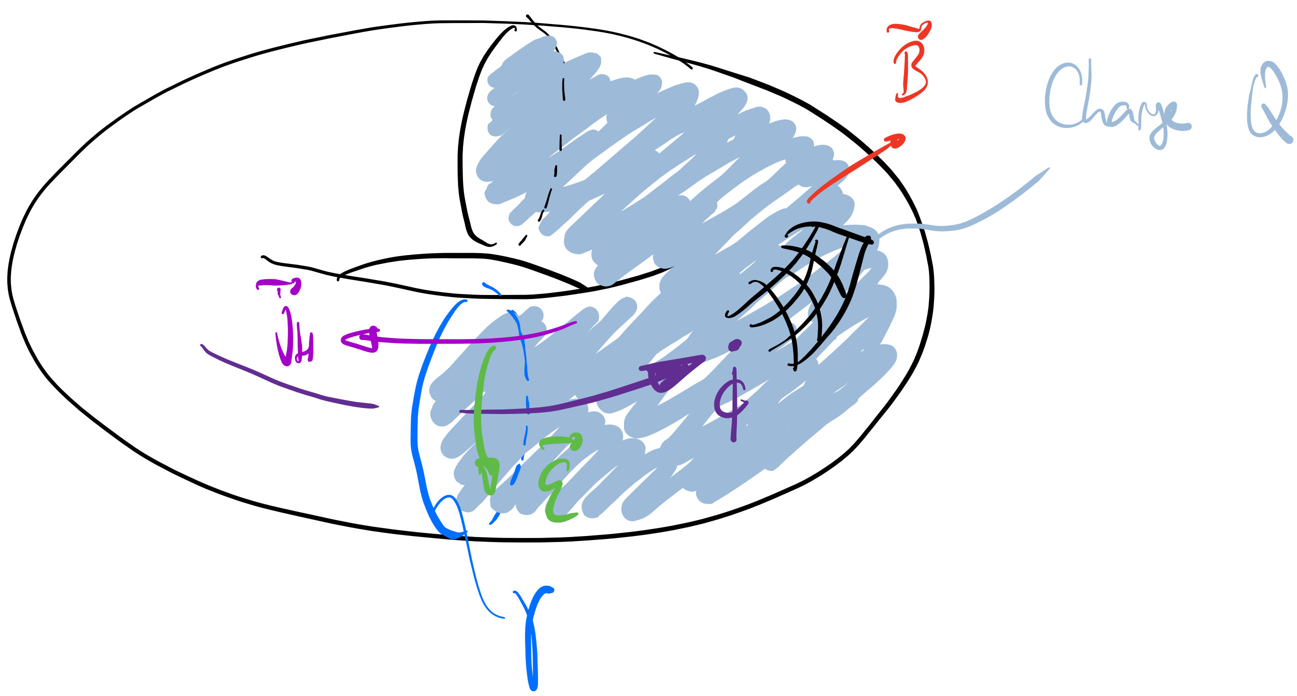
Laughlin's argument:



Laughlin revisited :



Many-body setting :



Classically : $m\dot{v} = q\left(E + \frac{v}{c} \wedge B\right) - \gamma v$

friction.

Stationary situation : $\dot{v} = 0$:

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \gamma/q & -B/c \\ B/c & \gamma/q \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Resistivity defined by : $E = \rho j$ ← current density

$$\rho = \frac{1}{qn} \begin{pmatrix} \gamma/q & -B/c \\ B/c & \gamma/q \end{pmatrix}$$

n: particle density , $j = q n v$

Conductivity

$$\sigma = \rho^{-1} = \frac{qn}{\gamma^2/q^2 + B^2/c^2} \begin{pmatrix} \gamma/q & B/c \\ -B/c & \gamma/q \end{pmatrix}$$

off-diagonal term σ_H Hall conduc.

diagonal term σ_D direct, dissipative

Why "dissipative" ?

dissipated power : $j \cdot E = \sigma E \cdot E$

only the symmetric part of

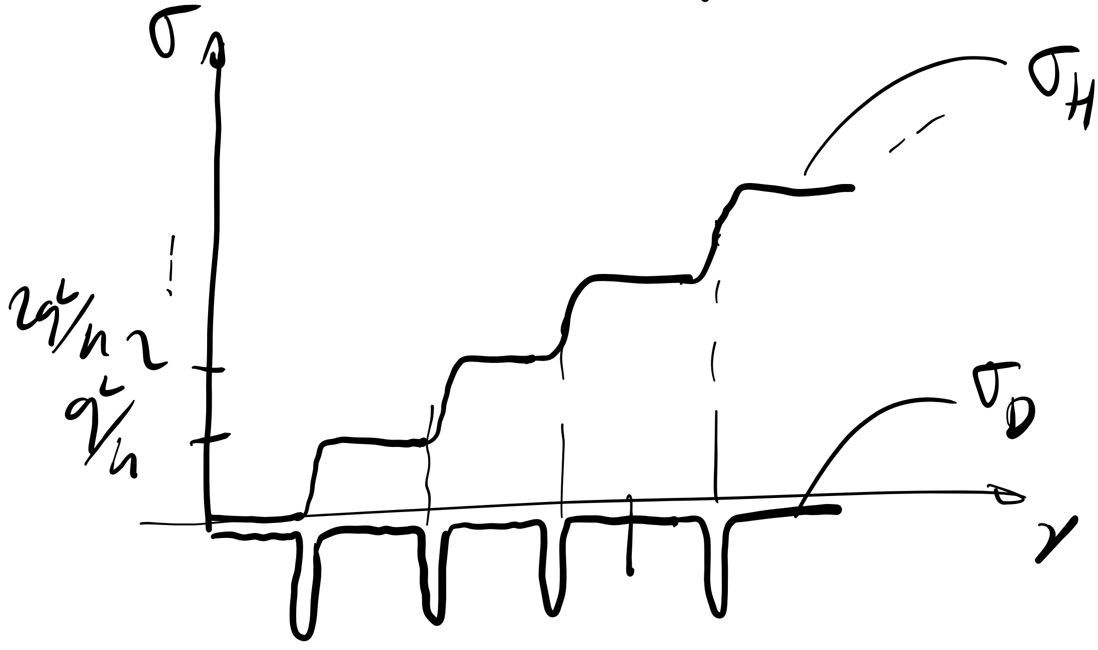
σ appears, namely σ_D

Ideal case $\gamma \to 0$, with $B \neq 0$

* $\sigma_D \to 0$
* $\sigma_H \to \dfrac{qnc}{B} = \operatorname{sgn}(q)\,\dfrac{q^2}{h}\,\nu$

where $\nu = \dfrac{nch}{|q|B}$ "filling factor"

Experiment (von Klitzing et al. 1980)



* Laughlin's argument

Faraday's law : $-\partial_t \Phi = c \oint_\gamma \vec{\varepsilon} \cdot d\vec{\ell}$

Hall current $j_H = -\sigma_H \, \vec{e}_r \wedge \vec{\mathcal{E}}$

So current across line γ

$$I = \int_\gamma j_H \cdot e_t \, dl = \sigma_H \int_\gamma \vec{\mathcal{E}} \cdot d\vec{l} = -\frac{\sigma_H}{c} \partial_t \Phi$$

Integrate:

$$\Delta Q = \sigma_H \frac{\Delta \Phi}{c}$$

In particular, if $\Delta \Phi = \frac{hc}{q}$ "flux quantum":

$$\frac{\Delta Q}{q} = \sigma_H \frac{h}{q^2}$$

Hall conductance = charge transported by a flux quantum.

Need to understand why quantized charge transport.

Microscopic Hamiltonian (Landau Ham.)

$$H(\phi) = \frac{1}{2m}\left(p - \frac{q}{c} A(\phi)\right)^2$$

where $A(\phi) = A_B + A_f(\phi) = \begin{pmatrix} 0 \\ Bx_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\phi}{2\pi R} \end{pmatrix}$

$$H(\phi) = \frac{1}{2m}\left(p_1^2 + \left(p_2 - \frac{q}{c}\left(Bx_1 + \frac{\phi}{2\pi R}\right)\right)^2\right)$$

Now $[H(\phi), p_2] = 0$; moreover: periodic BC imply $e^{2\pi i R \ell} = 1$ for any eigenvalue ℓ of p_2, namely $2\pi R\ell = 2\pi n$, $n \in \mathbb{Z}$

Since p_2 is conserved, we look at $H_n(\phi)$, where p_2 is replaced by $\frac{n}{R}$

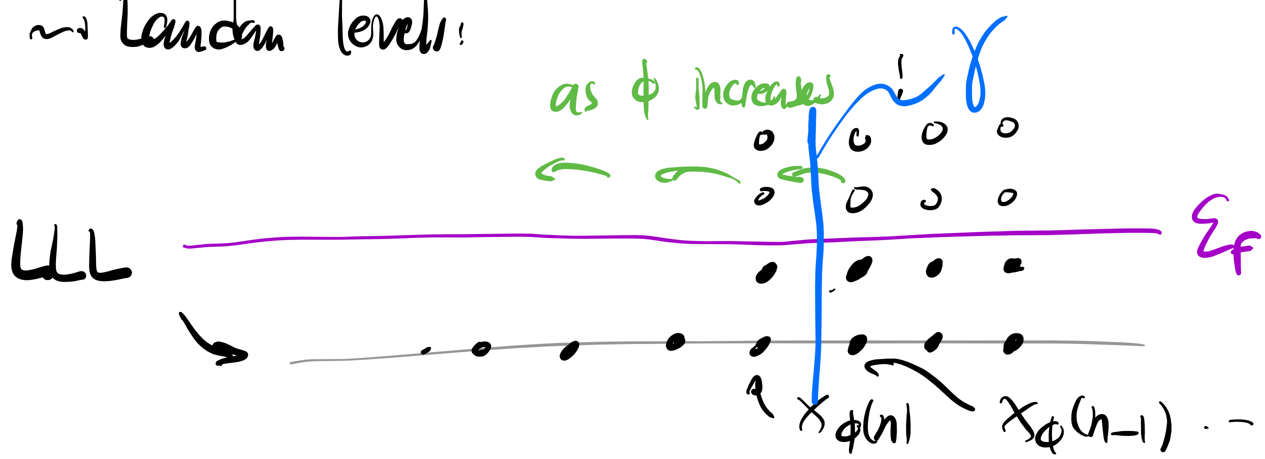
$$H_n(\phi) = \dots = \frac{1}{2m}\left(p_1^2 + \left(\frac{qB}{c}\right)^2\left(x_1 - x_\phi(n)\right)^2\right)$$

$$x_\phi(n) = c\left(n\frac{hc}{q} - \phi\right)$$

Harmonic oscillator centered at $x_1 = x_\phi(n)$

Spectrum of $H_n(\phi)$:

$\rightsquigarrow$ Landau levels:



As Φ increases by $\frac{hc}{q}$

$$n\frac{hc}{q} - \Phi - \frac{hc}{q} = (n-1)\frac{hc}{q} - \Phi \quad \text{namely}$$

$$\frac{\Delta q}{q} = (\#\text{ of occupied Landau level})$$
$$= \sigma_H \quad \text{in particular: integer. :)}$$

Avron - Seiler - Simon '94 : Rigorous version of

Compare Fermi projection P of the physical system (e.g. Landau Hamiltonian on $L^2(\mathbb{R}^2)$) with $UPU^* =: Q$ for a well-chosen U

For $\dim P < \infty$, say $\dim P = N$, then N is the number of electrons in the system and

$$\Delta N = \mathrm{Tr}(P) - \mathrm{Tr}(UPU^*)$$
$$= \mathrm{Tr}(P) - \mathrm{Tr}(P) = N - N = 0$$

$\rightsquigarrow$ Consider thermodynamic limit at finite density i.e. "$N = \infty$" ; $\dim P = \infty$

"Fredholm index"

- Two proj. P, Q. Let

$C = P - Q$; $S = P + Q - \mathbb{1} = P - Q^\perp$

Claim: * $C^2 + S^2 = \mathbb{1}$

* $CS + SC = 0$

Indeed: * $C^2 = P - PQ - QP + Q$

$= PQ^\perp + QP^\perp$

S^2: same with $Q \leftrightarrow Q^\perp$

Add the two, use that $Q + Q^\perp = \mathbb{1}$

to get $P + P^\perp = \mathbb{1}$.

* $CS = P - PQ^\perp - QP = + PQ - QP$

is antisymmetric.

Now: Assume $P - Q$ is compact.

Hence $\mathrm{Spec}(C)$: $\lambda_0 \geq \lambda_1 \geq \dots \geq$

the eigenvalues can accumulate only at 0.

Let ψ : $C\psi = \lambda\psi$

Then $\underset{\lambda S\psi}{\underset{\parallel}{SC\psi}} = -CS\psi$

Hence if $S\psi \neq 0$, then $-\lambda$ is also in $\mathrm{Spec}(C)$

if $S\psi = 0$:

$$0 = \|S\psi\|^2 = \langle \psi, (\mathbb{1} - C^2)\psi \rangle$$
$$= (1-\lambda^2) \qquad (\|\psi\|=1)$$

so $\lambda = \pm 1$

i.e. eigenvalues of C come in opposite pairs, except possibly ± 1, and 0

If $(P-Q)^{2n+1} \in \mathcal{I}_1$ (trace-class), then

$$\mathrm{Tr}\left((P-Q)^{2n+1}\right) = \sum_j \lambda_j^{2n+1}$$
$$= \dim \mathrm{Ker}\,(P-Q-\mathbb{1})$$
$$- \dim \mathrm{Ker}\,(P-Q+\mathbb{1})$$

And in fact: the trace is independent of n, provided $(P-Q)^{2n+1} \in \mathcal{I}_1$

$$\Rightarrow \mathrm{Ind}(P,Q) := \dim \mathrm{Ker}\,(P-Q-\mathbb{1}) - \dim \mathrm{Ker}\,(P-Q+\mathbb{1})$$

Example $\mathcal{H} = l^2(\mathbb{Z})$ $P = \chi_{\{\mathbb{N}\}}$

U is the shift

$$(U\psi)(j) = \psi(j-1)$$

$P - UPU^* =$ projection onto 1-d span of $(\psi^{(0)})(j) = \delta_{0,j}$

$$\mathrm{Tr}(P - UPU^*) = 1$$

<u>Remarks</u>: + This is a local contribution: P and UPU^* differ only "at the cut"

+ Obviously $\mathrm{Ind}(P, Q) \in \mathbb{Z}$

+ $\mathrm{Ind}(P,Q)$ can be phrased as a Fredholm index. Let $T = PU^*P$. Then

$$T^*T = PQP \quad ; \quad TT^* = U^* UPU^*PUPU^*U = U^*QPQU$$

and so

$\dim \mathrm{Ker}_P T^*T - \dim \mathrm{Ker}_P TT^*$ (on range P)

$= \mathrm{Tr}(P - PQP) - \mathrm{Tr}(P - U^*QPQU)$

$= \mathrm{Tr}(PQ+P) - \mathrm{Tr}(UPU^* - QPQ)$

$$= \mathrm{Tr}(PQ^\perp P - QP^\perp Q)$$

and note that

$$PQ^\perp P = (P-Q)Q^\perp P = (P-Q)(1-Q)P$$
$$= (P-Q)^2 P$$

Hence: $QP^\perp P = (Q-P)^2 Q$ (exchanging $P \leftrightarrow Q$)

$$\dim \mathrm{Ker}_p(T^*T) - \dim \mathrm{Ker}_p(TT^*) =$$
$$\mathrm{Tr}\left((P-Q)^2 \cdot (P-Q)\right) = \mathrm{Tr}\left((P-Q)^3\right)$$
$$= \mathrm{Ind}(P,Q) \quad \text{indeed.}$$

Note: $\mathrm{Ker}\, T^*T = \mathrm{Ker}\, T \cup (\underbrace{\mathrm{Ker}\, T^*}_{=(\mathrm{Ran}(T))^\perp} \cap \mathrm{Ran}(T))$
$$= \mathrm{Ker}\, T$$

We conclude:

$$\mathrm{Ind}(P,Q) = \dim \mathrm{Ker}(T) - \dim \mathrm{Ker}(T^*) = \mathrm{Ind}(T)$$

by definition: the Fredholm index of $T = PU^*P$.

Crucial properties:

(i) $\mathrm{Ind}(P,Q) = \mathrm{Ind}(\tilde{U}P\bar{U}, \tilde{U}Q\hat{U})$

(ii) $\mathrm{Ind}(P,R) = \mathrm{Ind}(P,Q) + \mathrm{Ind}(Q,R)$

In particular: we write $\mathrm{Ind}(P,U) = \mathrm{Ind}(P, UPU^*)$

$$\mathrm{Ind}(P, U_1U_2) = \mathrm{Ind}(P, U_1) + \mathrm{Ind}(P, U_2)$$

Results: Avron-Seiler-Simon

(i) Charge transport $= -\mathrm{Index}\left(P \frac{z}{|z|} P\right)$

here $z = x+iy$ and $\frac{z}{|z|}$: multiplication operator on $L^2(\mathbb{R}^2)$

(ii) If P_n is the projection onto the n^{th} Landau level, then

$$\mathrm{Ind}\left(P_n \frac{z}{|z|} P_n\right) = -1$$

Technicalities: * P_n has integral kernel $p_n(x,y)$ with sufficient decay:

$$|p_n(x,y)| \leq \frac{C}{1+|x-y|^\eta} \qquad \eta > 2$$

in fact: ~~for~~ P_n, the decay is even exponential

We have $(P-Q)^3$ is trace-class

* Explicit calculation of

$$\left(P_n \frac{z}{|z|} P_n\right)_{j,j'} = c(n,j)\,\delta_{j,j'+1} \quad \text{essentially a shift.}$$

Part 2: (Quantized) quantum transport for interacting, many-body systems. (with Bols, De Roeck, Fraas)

- Setting: Lattice system: either spin or fermions.
  In both cases on $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^2$
  Important: all constants indep. of L
  - fermions: 1-part. Hilbert space $\ell^2(\Lambda_L)$
    Algebra of observables $CAR(\ell^2(\Lambda_L))$,
    generated by a_x^*, a_x, $x \in \Lambda_L$
    $a_x^* = a^*(\delta_x)$

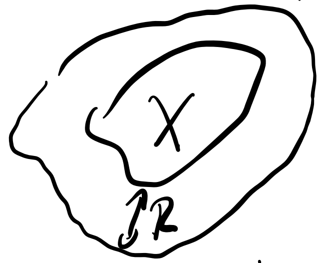
  Denote $\hat{\mathcal{A}}_X = CAR(\ell^2(X))$ $\quad X \subset \Lambda_L$
  Use $\mathcal{A}_X$: made of <u>even</u> observables
  $\hookrightarrow$ $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $X \cap Y = \emptyset$
  then $[A, B] = 0$

  - Quantum spin, see Bruno's lecture.

- In both cases: we have an interaction $\Phi(X)$
  hence a Hamiltonian $H = \sum_{X \subset \Lambda_L} \Phi(X)$

Key locality estimate. $t>0$, $A \in \mathcal{A}_X$.
Then for any $R \geq 0$, $\exists\ \tilde{A}_R(t)$ s.t.

* $\tilde{A}_R(t) \in \mathcal{A}_{X(R)} \leftarrow = \{x \in \Lambda_L : \mathrm{dist}(x, X) \leq R\}$



* $\| e^{itH} A e^{-itH} - \tilde{A}_R(t) \| \leq C(A) e^{vt} \zeta(R)$

where ζ is a non increasing, positive function with fast decay: $\sup_R R^n \zeta(R) < \infty$ $\forall n \in \mathbb{N}$

* Observables "well-localized" around X: $\tilde{\mathcal{A}}_X$.

$B \in \tilde{\mathcal{A}}_X$ if $\exists\ B_n \in \mathcal{A}_{X(n)}$ s.t.

$$\|B\| + \sup_n \frac{\|B - B_n\|}{\zeta(n)} < \infty$$

* We will need:

(i) A projection P with $\dim(P) = p < \infty$

e.g. Free Fermi gas: $p=1$ and P is the proj. onto a Slater determinant.

(ii) A charge operator Q_Z, $Z \subset \Lambda_L$

$$Q_Z = \sum_{x \in Z} q_x \qquad \text{with}$$

* $[q_x, q_y] = 0$ if $x \neq y$
* $\mathrm{Spec}(q_x) \subset \mathbb{Z}$

$$\Rightarrow \mathrm{Spec}(Q_Z) \subset \mathbb{Z}$$

Example : $q_x = a_x^\dagger a_x$

(iii) $\exists$ unitary U (e.g. flux insertion)

* has a local action :

$$A \in \mathcal{A}_X \Rightarrow U^\dagger A U \in \widetilde{\mathcal{A}}_X$$

Examples: e^{itH}, or translation by a lattice vector.

* Charge conservation: $U^\dagger Q_{\Lambda_L} U - Q_L = 0$

or $[U, Q_{\Lambda_L}] = 0$

Note: $Z \subset \Lambda_L$: $0 = [U, Q_Z] + [U, Q_{Z^c}]$

with $[U, Q_Z] \in \widetilde{\mathcal{A}}_Z$

$$\Rightarrow [U, Q_Z] \in \widetilde{\mathcal{A}}_Z \cap \widetilde{\mathcal{A}}_{Z^c}$$

Hence $[U, Q_Z] \in \widetilde{\mathcal{A}}_{\partial Z}$

where $\partial Z = \{x \in \Lambda_2 : \operatorname{dist}(x, Z) \leq 1$
$\operatorname{dist}(x, Z^c) \leq 1\}$.

- Now: $Q := Q_{[0, \frac{L}{2}] \times [0, L]}$ which define two boundaries, hence:

$$U^\dagger Q U - Q = T_- + T_+ \quad \text{with } T_\pm \in \tilde{\mathcal{A}}_{\partial_\pm}$$

⚠ Total charge always conserved so we concentrate on transport across ∂_-, given by T_-.

Example: U: translation by one site

$$U^\dagger Q U = Q_{[1, \frac{L}{2}+1] \times [0, L]}$$

$$T_- = Q_{[1]}$$

Note: choice of $T_\pm$ is not unique.

We shall fix $T_\pm$ s.t.

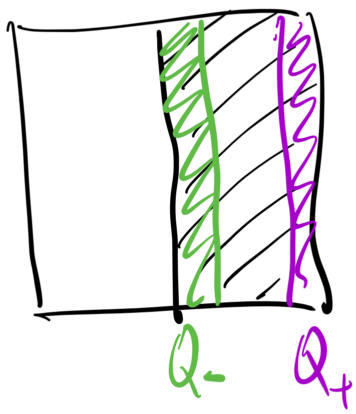
$$e^{2\pi i (Q + T_-)} = \mathbb{1}.$$

Lemma: Such a T_- exists.

Proof: $\mathrm{Spec}(Q) \subset \mathbb{Z}$ so

$$\mathbb{1} = e^{2\pi i U^* Q U} = e^{2\pi i (Q + T_- + T_+)}$$

Split: $Q = Q_- + Q_m + Q_+$



Strip of order L

because of the tails of $T_\pm$.

Hence $e^{2\pi i (Q + T_- + T_+)} =$

$$e^{2\pi i (Q_- + T_-)} \underbrace{e^{2\pi i Q_m}}_{= \mathbb{1}} e^{2\pi i (Q_+ + T_+)} + O(L^{-\infty})$$

$$\Rightarrow \mathbb{1} \cong \underbrace{e^{2\pi i (Q_- + T_-)}}_{\in \mathcal{A}_{\partial_-}} \underbrace{e^{2\pi i (Q_+ + T_-)}}_{\in \mathcal{A}_{\partial_+}}$$

$$\Rightarrow e^{2\pi i (Q_- + T_-)} \cong e^{2\pi i \nu} \mathbb{1} \qquad \nu \in [0,1]$$

Conclude by redefining $T_- \to T_- - \nu$

$$e^{2\pi i(Q_- + T_-)} \cong \mathbb{1}$$

$$\overset{\parallel}{e^{2\pi i(Q + T_-)}} \quad (\text{add } Q_m, Q_+ \text{ for free})$$

$\boxtimes$

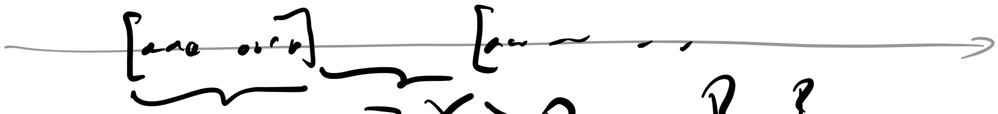
- Assumptions :
  - Pumping , $\| [U, P] \| = O(L^{-\infty})$ (see Laughlin's argument) (cyclic process)
  - Clustering : $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $d = \mathrm{dist}(X, Y)$, $\|A\|, \|B\| = 1$

    For any $\Omega = P\Omega$

    $$\left| \langle \Omega, AB\Omega \rangle - \langle \Omega, APB\Omega \rangle \right| = |X||Y| O(d^{-\infty})$$
  - Local charge fluctuations (LCF) :
    $\exists\, \bar{Q}$ s.t.
    - $\bar{Q} = Q + K_- - K_+ \qquad K_\pm \in \tilde{\mathcal{A}}_{\partial_\pm}$
    - $[\bar{Q}, P] = 0$

Rem : (ii) can be proved if P is a spectral projection for a local Hamiltonian, with a spectral gap.

Spec(H)



interval I $\geq \gamma > 0$ $P = P_I$.

(iii) can be proved in the same situation, if
$$H = \sum_X \Phi(X) \text{ with } [\Phi(X), Q_{\Lambda_L}] = 0$$

Theorem: Consider the state $\omega_p(\cdot) = \frac{1}{p} \mathrm{Tr}(P(\cdot))$.

Then $\omega_p(T_-) \in \frac{\mathbb{Z}}{p}$ (up to $\mathcal{O}(L^{-\infty})$)

Transport is quantized, but now in rationals.

denominator is $\dim(P)$

expected indeed for fractional quantum Hall effect

Note: If P is a Slater determinant (i.e. non-interacting setting) then $p=1$ and we recover an integer.

Proof: $$Z(\phi) = \underbrace{U^* e^{i\phi \bar{Q}} U}\, e^{-i\phi \bar{Q}} \qquad \phi \in [0, 2\pi)$$

Note: $Z(\phi) = Z_-(\phi) Z_+(\phi)$

where:

$$z_-(\phi) = e^{i\phi \bar{Q}_-^U} e^{-i\phi \bar{Q}_-} \in \mathcal{A}_{\partial_-}$$

Where: $\bar{Q}_- = Q_- - K_-$

$\bar{Q}_-^U = Q_- - T_- - U^\dagger K_- U$

why? $z(\phi) = e^{i\phi(Q + T_+ + T_- - U^\dagger K_- U - U^\dagger K_+ U)} e^{-i\phi(Q - K_- - K_+)}$

Split $Q = Q_- + Q_m + Q_+$ and note the Q_m's cancel out

Now: $[z(\phi), P] \cong 0$

Claim: $[z_-(\phi), P] \cong 0$

Lemma: Unitary V, P. Then

(i) $\| [V, P] \| < \varepsilon$

$\Rightarrow \| P V \psi \|^2 > 1 - \varepsilon \qquad \psi = P\psi$

$\| P V^\dagger \psi \|^2 > 1 - \varepsilon$

(ii) $\| P V \psi \|^2 > (1 - \varepsilon^2) \| P\psi \|^2$

and $\| P V^\dagger \psi \|^2 > (1 - \varepsilon^2) \| P\psi \|^2$

$\Rightarrow \| [V, P] \| < 2\varepsilon$

Proof: (i) $\| P V \psi \|^2 = \langle \psi, V^\dagger [P, V] \psi \rangle + \langle \psi, P\psi \rangle$

$$> 1-\varepsilon$$

(ii) For any ψ:

$$\|P\psi\|^2 = \|VP\psi\|^2$$
$$= \|P^\perp VP\psi\|^2 + \underbrace{\|PVP\psi\|^2}_{> (1-\varepsilon^2)\|P\psi\|^2}$$

$$\Rightarrow \|P^\perp VP\psi\|^2 < \varepsilon^2 \|P\psi\|^2$$

Repeat with $V \leftrightarrow V^*$:

Conclude with

$$[V, P] = P^\perp VP - PVP^\perp$$

$\blacksquare$

In our setting: $\quad Z = Z_- Z_+$

We claim $[Z, P] \simeq 0 \Rightarrow [Z_-, P] \simeq 0$

Indeed:

$$[Z, P] \simeq 0 \Rightarrow \|PZ_+Z_-\Omega\|^2 \simeq 1$$

By clustering: $\|PZ_+Z_-\Omega\| \simeq \|PZ_+PZ_-\Omega\|$

Hence: $\quad 1 \simeq \|PZ_+PZ_-\Omega\|$
$$\leq \|PZ_+\|\,\|PZ_-\Omega\| = \|PZ_-\Omega\| \leq 1$$

$\Rightarrow \|PZ_-\Omega\| \simeq 1$ and similarly with Z_-^*.

Conclude by lemma: $\| [P, Z_-] \| \simeq 0$

Note: also: $\left[e^{2\pi i \bar{Q}_-}, P \right] = 0$ (only at 2π)

Consider: $\chi(\phi) = P Z_-(\phi) P$

denote: $D_- = \bar{Q}_-^U - \bar{Q}_-$

$$\Rightarrow \quad -i \partial_\phi \chi(\phi) = P e^{i\phi \bar{Q}_-^U} (D_-) e^{-i\phi \bar{Q}_-} P$$

$$= P \chi(\phi) e^{i\phi \bar{Q}_-} D_- e^{-i\phi \bar{Q}_-} P$$

$$= P \chi(\phi) e^{i\phi \bar{Q}} D_- e^{-i\phi \bar{Q}} P$$

$$-i \partial_\phi \chi(\phi) = \chi(\phi) e^{i\phi \bar{Q}} P D_- P e^{-i\phi \bar{Q}}$$

Solution: $\chi(\phi) = e^{i\phi (P D_- P - \bar{Q})} e^{i\phi \bar{Q}}$ $\qquad (*)$

At $\phi = 2\pi$:

$$\overline{U^\dagger e^{2\pi i \bar{Q}_-}} U = U^\dagger e^{2\pi i (\bar{Q}_- + Q_m + Q_+)} U$$

$$= e^{2\pi i (U^\dagger Q U - U^\dagger K_- U)}$$

$$= \underbrace{e^{2\pi i (Q_+ + T_+)}}_{= \mathbb{1} \text{ (by choice of } T_+)} e^{2\pi i (Q_- + T_- - U^\dagger K_- U)}$$

$$= e^{2\pi i \bar{Q}_-^U}$$

$$\rightarrow \chi(2\pi) = P \underbrace{U^\dagger e^{2\pi i \bar{Q}_-} U} e^{-2\pi i \bar{Q}_-} P$$

Now: All 4 unitary matrices commute with P

Take determinant on $\mathrm{Ran}(P)$:

$$\det_P \chi(2\pi) = 1$$

Also: from $(*)$:

$$\det \chi(\phi) = e^{i\phi \operatorname{Tr}(P D_- P - \cancel{\bar{Q}})} e^{i\phi \operatorname{Tr}(\cancel{\bar{Q}})}$$

$$\left(\det e^A = e^{\operatorname{Tr} A}\right)$$

$$= e^{i\phi \operatorname{Tr}(P D_- P)}$$

$$\rightarrow e^{2\pi i \operatorname{Tr}(P D_- P)} = 1 \quad \text{(by above)}$$

Finally: $\operatorname{Tr}(P D_- P)$

$$= \operatorname{Tr}\left(P\left(\bar{Q}^U_- - \bar{Q}_-\right)P\right)$$

$$= \operatorname{Tr}\left(P\left(T_- - U^\dagger K_- U + K_-\right)P\right)$$

$$= \operatorname{Tr}(P T_- P) + \underbrace{\operatorname{Tr}\left(P(K_- - U^\dagger K_- U)P\right)}_{\substack{=0 \text{ by } [U,P]=0 \\ \text{and cyclicity}}}$$

We conclude:

$$e^{2\pi i \underbrace{\mathrm{Tr}(PT_-P)}_{=\mathrm{Tr}(PT_-)}} \approx 1$$

$$\Rightarrow \mathrm{Tr}(PT_-) \in \mathbb{Z} \qquad \blacksquare$$

- **Property**: Denote $\mathrm{Ind}_P(U) = ar_p(T_-)$.

Then $\mathrm{Ind}_p(U_1U_2) = \mathrm{Ind}_p(U_1) + \mathrm{Ind}_p(U_2)$

Proof: follows from.

$$T(U_2U_1)_- = U_2^* T(U_1)_- U_2 + T(U_2)_- \qquad \square$$

Auxiliary results:

① **LCF**. For any $g \in L^1(\mathbb{R})$, define

$$I_g(A) = \int g(t)\, e^{itH} A e^{-itH}\, dt$$

Pick W s.t. (i) $\sup_t |t|^n |W(t)| < \infty$ $\forall n$

(ii) $\hat{W}(\xi) = \frac{i}{\sqrt{2\pi}\,\xi}$ for $|\xi| \geq \gamma$.

Here: γ is the gap of the local Hamiltonian H for which P is a spectral projection.

Should have said: H conserves charge:

$$[H, Q_{\Lambda}] = 0$$

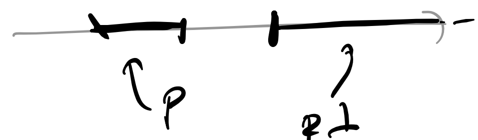
Prop: $\left[I_W\left(i[H,A]\right), P \right] = [A, P] \quad \forall A.$

Proof: By spectral theorem:

$$I_W\left(i[H,A]\right) = \int W(t)\, e^{it(\lambda-\mu)}\, dP_\lambda\, i[H,A]\, dP_\mu\, dt$$

$$= \sqrt{2\pi}\, i \int \hat{W}(\mu-\lambda)\,(\lambda-\mu)\, dP_\lambda\, A\, dP_\mu$$

Now: Since $P[A,P]P = 0$



$P^\perp [A,P] P^\perp = 0$

In other words: $[A,P]$ is off-diagonal with respect to

$$\mathcal{H} = P\mathcal{H} \oplus P^\perp \mathcal{H}$$

Because of that, the integral reduces to the domain $|\lambda-\mu| > \gamma$, where $\hat{W}(\mu-\lambda) = \frac{i}{\sqrt{2\pi}(\mu-\lambda)}$

So:

$$\left[I_W\left(i[H,A]\right), P \right] = \int_{|\lambda-\mu|>\gamma} dP_\lambda [A,P]\, dP_\mu = [A,P].$$

$\square$

Note: We have shown: For any off diag.

operator : $\mathcal{I}_W(i[H,A]) = A$

or $i[H, \mathcal{I}_W(A)] = A$

i.e. $\mathcal{I}_W(\cdot)$ is the inverse of $i[H,\cdot]$ on the set of off-diagonal op.

Apply to Q: $[Q,P] = [\mathcal{I}_W(i[H,Q]), P]$

$\rightarrow \overline{Q} = Q - \mathcal{I}_W(i[H,Q])$

Locality of $\mathcal{I}_W(i[H,Q])$?

For a charge conserving Hamiltonian:

$$i[H,Q] = J_- + J_+$$

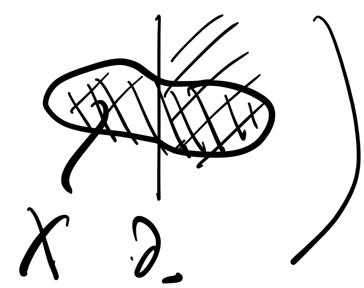
$$J_\pm \in \mathcal{A}_{\partial_\pm}$$

($H = \sum \Phi(X)$, and : $[\Phi(X), Q]$

whenever $X \subset [0,\frac{L}{2}] \times [0,L)$

or $X \subset ([0,\frac{L}{2}] \times [0,L))^c$

so $[H,Q] = \sum_{\substack{X:\\ X\cap [0,\frac{L}{2}]\times[0,L) \\ \text{and } X\cap([0,\frac{L}{2})\times[0,L))^c}} [\Phi(X),Q]$

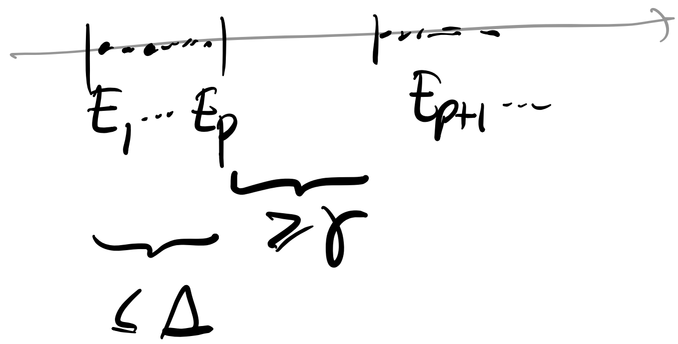


Finally: $\mathcal{I}_W(J_-) \in \widetilde{\mathcal{A}}_{\partial_-}$

by (i) fast decay of W (large times in integral)
(ii) Lieb-Robinson bound (short times)

$\square$

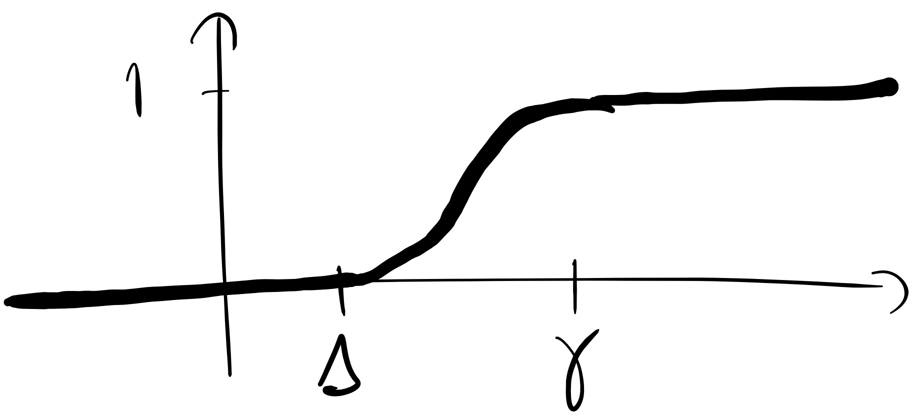
② Clustering $\mathrm{Spec}(H)$:
(see also Nachtergaele-Sims, Hastings-Koma)



Assumption $\Delta < \gamma$.

We consider g s.t. $g(t) = O(|t|^{-\infty})$ and

$$\hat{g}(\xi) = \begin{cases} 0 & \text{for } \xi < \Delta \\ 1/\sqrt{2\pi} & \text{for } \xi \geq \gamma \end{cases}$$



Claim:
$$\begin{cases} (i) \; I_g(A) P = 0 \\ (ii) \; P I_g(A) = P A P^{\perp} \end{cases}$$

Indeed: $I_g(A) = \int g(t) e^{itH} A e^{-itH} dt$

$$= \sum_{i,j=1}^{d} \sqrt{2\pi}\, \hat{g}(E_i - E_j)\, P_j A P_i$$

(i) : Multip. on right by P restricts the sum to $i = 1, \dots, p$

Thus: $E_i - E_j < E_p - E_1 < \Delta$

$\forall i,j$ in the sum.

Hence $\hat{g}(E_i - E_j) = 0 \quad \forall i,j$ in sum.

$$\Rightarrow I_g(A) P = 0$$

(ii) By (i) $\quad P I_g(A) = P I_g(A) P^\perp$

$$= \sum_{j=1}^{p} \sum_{i=p+1}^{d} \sqrt{2\pi}\, \hat{g}(E_i - E_j)\, P_j A P_i$$

here: $E_i - E_j > E_{p+1} - E_p \geq \gamma$

therefore: $\hat{g}(E_i - E_j) = \frac{1}{\sqrt{2\pi}} \quad \forall i,j$ in sum.

$$\Rightarrow \quad P I_g(A) = P A P^\perp$$

Now: $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $d(X, Y) = d > 0$

$$PABP = PAPBP + PAP^\perp BP$$

$$\overset{(ii)}{=} PAPBP + P I_g(A) BP$$

$\overset{(i)}{\Rightarrow}$ $PAPBP + P[\mathcal{I}_g(A), B]P$

Since $A \in \mathcal{A}_X$, $\mathcal{I}_g(A) \in \widetilde{\mathcal{A}}_X$, hence

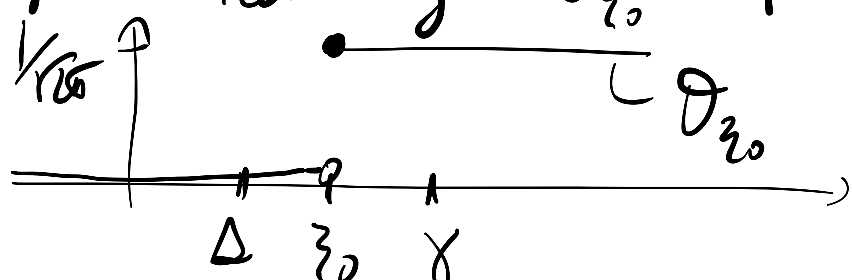
$$\|[\mathcal{I}_g(A), B]\| = C(A,B)\, O(d^{-\infty})$$

$$\Rightarrow \quad \|PABP - PAPBP\| = O(d^{-\infty}) \qquad \square$$

Note: In the last part, the Fourier transform must be understood in the sense of distributions on S'.

Concretely: Pick $\hat{g} = \theta_{\xi_0} * \varphi$

where



$\xi_0 = \frac{1}{2}(\gamma+\Delta)$

and $\varphi \in C_c^\infty(-\delta, \delta)$, $\delta = \frac{1}{2}(\gamma - \Delta)$

This yields

principal part

$$g(t) = \left(c_1 \delta(t) + i c_2\, \mathcal{P}\!\left(\frac{1}{t}\right) \right) e^{-i\xi_0 t} \check{\varphi}(t)$$

The singular support is $\{0\}$.

The regular part has fast decay, since φ is smooth.

- A simple application. Bloch's theorem, in any dimension

Here pick $U = U(t) = e^{itH}$

and $P = |\Omega\rangle\langle\Omega|$ where

Ω is the unique gapped G.S. of H.

in part. $[U, P] = 0$

Hence: $\langle \Omega, (U(t)^* Q U(t) - Q)_- \Omega \rangle \simeq \frac{n}{p}$ $\forall t$

$$\left(U(t)^* Q U(t) - Q\right)_- = \int_0^t \Big(U(s)^* \underbrace{i[H, Q]}_{= J_- + J_+} U(s)\Big)_- ds$$

$$= \int_0^t U(s)^* J_- U(s) ds$$

$\Rightarrow \quad \frac{n}{p} = \langle \Omega, J_- \Omega \rangle \cdot t$

$\Rightarrow \langle \Omega, J_- \Omega \rangle = 0$ i.e. $n = 0$

Absence of stationary current in G.S.